301. (a) $\mathbb{P}(\text{WW}) = \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$,

(b) There are two outcomes (BW and WB) which make up this event, so the probability is

$$\begin{aligned}\mathbb{P}(\text{one white, one black}) &= 2 \times \tfrac{2}{4} \times \tfrac{2}{3} \\ &= \tfrac{2}{3}.\end{aligned}$$

302. Division by $x$ destroys the possibility that $x$ could be zero. This can either be noted explicitly, adding $x = 0$ to the list of roots, or else the equation can be factorised as follows:

$$\begin{aligned}&x^3 = x \\ \Longrightarrow\ &x^3 - x = 0 \\ \Longrightarrow\ &x(x+1)(x-1) = 0 \\ \Longrightarrow\ &x = 0, \pm 1.\end{aligned}$$

303. We know that

$$|a| = |b| \iff a^2 = b^2.$$

So, we solve $(3x-1)^2 = (2-5x)^2$:

$$\begin{aligned}&9x^2 - 6x + 1 = 25x^2 - 20x + 4 \\ \Longrightarrow\ &16x^2 - 14x + 3 = 0 \\ \Longrightarrow\ &(8x-3)(2x-1) = 0 \\ \Longrightarrow\ &x = \tfrac{3}{8}, \tfrac{1}{2}.\end{aligned}$$

304. Substituting for $y$ gives $(x+1)^2 = 0$. The double root means that the statement is true: the curves are indeed tangent to each other at $x = -1$.

— Nota Bene —

To understand the significance of the double root (i.e. the squared factor), consider the displacement of the first curve above the second. This is

$$\begin{aligned}y_1 - y_2 &= x^2 - (-x^2 - 4x - 2) \\ &\equiv 2x^2 + 4x + 2 \\ &\equiv 2(x+1)^2.\end{aligned}$$

Since this displacement is always positive, the curves cannot cross. They do touch, however, at the double root $x = -1$.

305. The constant term $c$ must be 0, since $(0, 0)$ satisfies the equation. The other two coefficients cannot be determined.

306. (a) The equation $x = (y-1)^2 - 2$ is a positive quadratic with a vertex at $(-2, 1)$. This could be the equation of the parabola shown.

(b) $x = (y+1)^2 - 2$ has a vertex at $(-2, -1)$: no.

(c) $x = (y-1)^2 + 2$ has a vertex at $(2, 1)$: no.

307. Using the sine rule, labelling $\angle BCA$ as $\theta$, we have

$$\frac{\sin 30^\circ}{2} = \frac{\sin\theta}{2\sqrt{3}}.$$

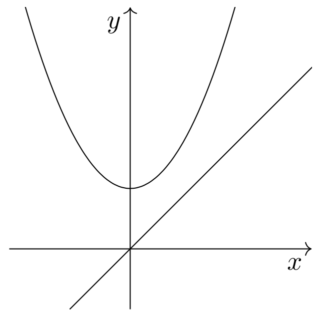
This gives $\theta = 60^\circ, 120^\circ$.

— Nota Bene —

This is known as the "ambiguous case" of the sine rule: there are two triangles, not congruent to each other, which fit the information given. It's a useful exercise to sketch them.

308. (a) Rearranging to $x^2 - x + 1 = 0$, the discriminant is $\Delta = 1 - 4 = -3 < 0$, so this quadratic has no real roots.

(b) The graph $y = x^2 + 1$ does not intersect $y = x$, and it is a positive parabola. So, it must be above the line $y = x$ everywhere.




Comparing the $y$ values above, we can see that, for every point on the parabola $y = x^2 + 1$, the output $y$ is greater than the input $x$. So, the instruction $x \mapsto x^2 + 1$ maps all real numbers to numbers larger than themselves.

309. Substituting the given values in, we get $4 + 2b = a$ and $12 - 4b = 6$. The latter equation gives $b = 3/2$. The former then gives $a = 7$.

310. Introducing a variable $y$ for the output,

$$\begin{aligned}&y = 4(x-2)^3 - 1 \\ \Longrightarrow\ &\tfrac{y+1}{4} = (x-2)^3 \\ \Longrightarrow\ &x = \sqrt[3]{\tfrac{y+1}{4}} + 2.\end{aligned}$$

Rewriting in terms of $x$, the inverse is

$$\mathrm{f}^{-1}(x) = \sqrt[3]{\tfrac{x+1}{4}} + 2.$$

311. The discriminants of the quadratic factors are $\Delta = 4, 0, -4$ respectively. So, (a) is false, because its quadratic has real roots, and (c) is true, because its quadratic has no real roots. In (b), a further consideration is required: the quadratic factorises as $(x-2)^2$, with one root at $x = 2$, hence, the implication also holds there.

312. Let $z = x^3$. This gives

$$\begin{aligned} & z^2 - 9z + 8 = 0 \\ \Longrightarrow\ & (z-1)(z-8) = 0 \\ \Longrightarrow\ & z = 1, 8. \end{aligned}$$

So, we have $x^3 = 1, 8$ and therefore $x = 1, 2$.

— Alternative Method —

Factorising directly as a quadratic in $x^3$,

$$\begin{aligned} & x^6 - 9x^3 + 8 = 0 \\ \Longrightarrow\ & (x^3-1)(x^3-8) = 0 \\ \Longrightarrow\ & x^3 = 1, 8 \\ \Longrightarrow\ & x = 1, 2. \end{aligned}$$

313. The operation of division by $x^2$, from the first to the second line, is undefined if $x^2 = 0$. The effect of this erroneous division by $x^2$ is to remove the possibility that $x^2 = 0$, thus deleting a root.

The second line should be a factorisation rather than a division:

$$\begin{aligned} & x^4 - x^2 = 0 \\ \Longrightarrow\ & x^2(x^2 - 1) = 0 \\ \Longrightarrow\ & x = 0, \pm 1. \end{aligned}$$

314. Differentiating f gives $f'(x) = \frac{4}{3}x^{-\frac{2}{3}}$. And the RHS of the differential equation is

$$\frac{f(x)}{3x} = \frac{4x^{\frac{1}{3}}}{3x} \equiv \tfrac{4}{3}x^{\frac{1}{3}-1} \equiv \tfrac{4}{3}x^{-\frac{2}{3}} = f'(x).$$

Since, upon substituting f in, the LHS and RHS are identically equal, $f(x)$ is a solution of the DE.

315. The large triangle is right-angled. Consider the gradients of the hypotenuse and the line segment joining $(5, 0)$ to $(4, 2)$. These are $-2$ and $1/2$, which are negative reciprocals. So, the line segment from $(5, 0)$ to $(4, 2)$ is perpendicular to the hypotenuse.

Hence, all three triangles have the same interior angles (AAA), and are therefore similar.

316. Evaluating the definite integral:

$$\begin{aligned} & 2\int_1^k t - 1 \, dt \\ \equiv\ & 2\Big[\tfrac{1}{2}t^2 - t\Big]_{-1}^{k} \\ \equiv\ & 2\Big(\big(\tfrac{1}{2}k^2 - k\big) - \big(\tfrac{1}{2} - 1\big)\Big) \\ \equiv\ & k^2 - 2k + 1 \\ \equiv\ & (k-1)^2, \text{ as required.} \end{aligned}$$

— Alternative Method —

Using the reverse chain rule, taking $t - 1$ to be the effective variable of integration,

$$\begin{aligned} & 2\int_1^k t - 1 \, dt \\ \equiv\ & \Big[(t-1)^2\Big]_1^k \\ \equiv\ & (k-1)^2, \text{ as required.} \end{aligned}$$

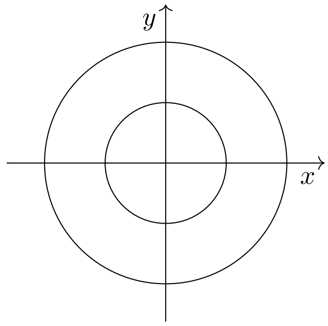
317. We rearrange to $x = y + 2$ and substitute, giving $(y+2)^2 - 4(y+2)$. This simplifies to $y^2 - 4$.

318. (a) These forces form a Newton pair: they are the two aspects of the single interaction between the elephants, according to NIII.

(b) The forces are equal in magnitude because of NII, as calculated for elephant $A$. The forces in question are the horizontal forces on elephant $A$: there is no acceleration, so these must be balanced.

319. (a) By the factor theorem, solution points require one of the brackets to be zero. Algebraically, this is $x^2 + y^2 = 1, 4$. These are the equations of two concentric circles.

(b) The circles are centred at the origin and have radii 1 and 2:




320. The area scale factor between A5 and A4 is 2, by definition. Hence, the length scale factor is $\sqrt{2}$. This means that the lengths of the sides of a sheet of A4 are in the ratio $1 : \sqrt{2}$.

321. Taking out a common factor of $(3x + 1)$, we have

$$(3x+1)\big((3x+1)(x+2)+2\big).$$

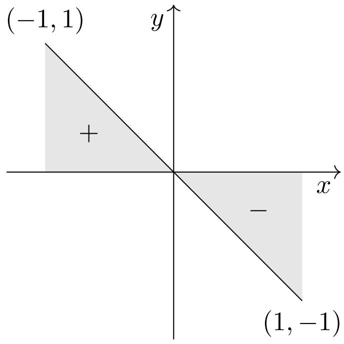
Since $x \in \mathbb{N}$, the factors are both natural numbers greater than 1. Their product cannot be prime.

322. (a) $6x^3 - 23x^2 + 24x - 7\Big|_{x=\frac{1}{2}} = 0.$

(b) By the factor theorem, $(2x - 1)$ is a factor of the cubic.

323. The interval contains all real numbers which are simultaneously larger than zero and less than or equal to 3. In interval set notation, this is $(0, 3]$.

324. Since the integral of f on $[-1, 1]$ is zero, the graph must be symmetrical about the origin; such is the only way in which the signed areas can cancel. Hence, we require




325. In each case, we multiply the number of paths on the way out by the number of available paths on the way back:

(a) $4 \times 4 = 16$,
(b) $4 \times 3 = 12$.

326. Multiplying top and bottom of the large fraction by the denominator of the inlaid fraction,

$$\begin{aligned} & \frac{1 + 1/x}{1 - 1/x} = x \\ \Longrightarrow\ & \frac{x+1}{x-1} = x \\ \Longrightarrow\ & x + 1 = x^2 - x \\ \Longrightarrow\ & x^2 - 2x - 1 = 0 \\ \Longrightarrow\ & x = 1 \pm \sqrt{2}. \end{aligned}$$

327. The toy is light, which means its mass is modelled as negligible, i.e. $m \approx 0$. So, it doesn't matter what acceleration the dogs are giving to the toy, NII dictates that the resultant force on the toy must be negligible. Assuming that no other forces are being applied to the toy, this is equivalent to the required result.

— Nota Bene —

In the question, there is no mention of the dogs or the toy being stationary. Hence, any explanation which uses equilibrium i.e. zero acceleration in its justification is not correct.

328. The 80 total marks must be scaled by $\frac{5}{4}$ to produce 100 total marks. So $M = \frac{5}{4}(X + Y)$.

329. Using the binomial expansion,

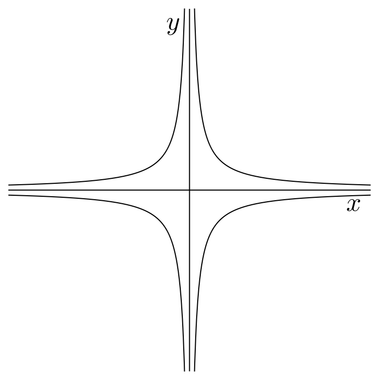
$$\begin{aligned} & \left(1 + \sqrt{3}\right)^5 \\ = {} & 1 + 5\sqrt{3} + 10\sqrt{3}^2 + 10\sqrt{3}^3 + 5\sqrt{3}^4 + \sqrt{3}^5 \\ = {} & 1 + 5 \cdot \sqrt{3} + 10 \cdot 3 + 10 \cdot 3\sqrt{3} + 5 \cdot 9 + 9\sqrt{3} \\ = {} & 76 + 44\sqrt{3}. \end{aligned}$$

330. The isosceles triangles along the top and bottom of the square have base 1 and height $1/2\sqrt{3}$. So, their combined area is $1/2\sqrt{3}$. Also, each of the equilateral triangles has area $\sqrt{3}/4$.

The shaded region is given by "the two equilateral triangles plus the two isosceles triangles minus the square." Evaluating this in terms of areas,

$$\begin{aligned} & 2 \cdot \tfrac{\sqrt{3}}{4} + \tfrac{1}{2\sqrt{3}} - 1 \\ = {} & \tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{3}}{6} - 1 \\ = {} & \tfrac{2\sqrt{3}}{3} - 1, \text{ as required.} \end{aligned}$$

331. Taking the square root, the equation is of the graph is $xy = \pm 1$. This is two reciprocal graphs, one positive, one negative:




332. $(6, 8, 10)$ is a Pythagorean triple, twice $(3, 4, 5)$. Its sine/cosine ratios are $3/5$ and $4/5$. The component of the particle's acceleration in the $\mathbf{j}$ direction is $4/5 \cdot 35 = 28 \text{ ms}^{-2}$.

— Nota Bene —

You can find angles explicitly here. However, it's often useful, when Pythagorean triples like $(3, 4, 5)$ are involved, to work directly with the trig ratios, as opposed to with the angles which produce those ratios. Here, angles are the long way round.

333. If the indefinite integral of f is quadratic, then f is a linear function $\text{f}(x) = ax + b$, where $a \neq 0$.

— Nota Bene —

The non-zero condition is required, otherwise there would be no $x^2$ term in the integral of $\text{f}(x)$.

334. (a) The circle has centre $(1, 1)$ and radius 1, so its equation is $(x - 1)^2 + (y - 1)^2 = 1$.

(b) To stretch by a factor 2 in the $x$ direction, we replace $x$ by $x/2$, giving $(x/2 - 1)^2 + (y - 1)^2 = 1$.

(c) To reflect in the line $y = x$, we switch variables $x$ and $y$, giving $(y/2 - 1)^2 + (x - 1)^2 = 1$.

335. The mean of an AP is the mean of the first and last terms. So $\frac{1}{2}(1+15)n = 64$. This gives $n = 8$, i.e. 7 steps. So the common difference is $\frac{15-1}{7} = 2$.

——— Alternative Method ———

Proceeding algebraically,

$$\begin{aligned} 15 &= 1 + (n-1)d, \\ 64 &= \tfrac{1}{2}n\big(2 + (n-1)d\big). \end{aligned}$$

The first equation is $(n-1)d = 14$. Substituting this into the second gives $n = 8$ and then $d = 2$.

336. (a) By Pythagoras, the squared distance between each pair of adjacent vertices is 26. Hence, the polygon is a rhombus.
(b) The diagonals have lengths $\sqrt{72}$ and $\sqrt{32}$. So, splitting into right-angled triangles, the area of the rhombus is $\frac{1}{2}\sqrt{72}\sqrt{32} = 24$.

337. This is a quadratic in $x^2$, so we can complete the square, treating $x^2$ as the variable. This gives

$$\mathrm{f}(x) = 19 - 16\big(x^2 - \tfrac{3}{4}\big)^2.$$

Since a square is always non-negative, this has maximum value 19.

——— Alternative Method ———

Setting the derivative to zero,

$$\begin{aligned} & 48x - 64x^3 = 0 \\ \Longrightarrow\ & x\big(48 - 64x^2\big) = 0 \\ \Longrightarrow\ & x = 0, \pm\tfrac{\sqrt{3}}{2}. \end{aligned}$$

Substituting these values back in,

$$\begin{aligned} \mathrm{f}(0) &= 10, \\ \mathrm{f}\left(\pm\tfrac{\sqrt{3}}{2}\right) &= 19. \end{aligned}$$

The function is a negative quartic, so its maximum must be among these values. Hence, the maximum value of $\mathrm{f}(x)$ is 19.

338. Since the events are pairwise mutually exclusive, they do not overlap, i.e. their intersections, both pairwise and as a three, must be empty. Hence, the probability of their union, which is the LHS, is given simply by the sum of their probabilities.

——— Alternative Method ———

The inclusion-exclusion principle states that

$$\begin{aligned} & \mathbb{P}(A \cup B \cup C) \\ =\ & \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ & - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) \\ & + \mathbb{P}(A \cap B \cap C). \end{aligned}$$

Setting the two-way and three-way intersections to zero gives the required result.

339. The sum of an AP of $n$ terms, whose first term is $a$ and whose common difference is $d$, is

$$S_n = \tfrac{1}{2}n\big(2a + (n-1)d\big).$$

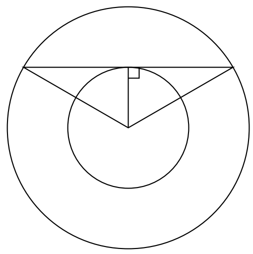
Applying this to the sequence of odd numbers,

$$\begin{aligned} S_{100} &= \tfrac{1}{2}100(2 + 99 \cdot 2) \\ &= 10000. \end{aligned}$$

340. The sum of the smaller forces is $2 + 5 = 7 < 8$. So, it is not possible to construct a triangle of forces for the object. Irrespective of the directions of the forces, the resultant force must have a non-zero component (magnitude $\geq 1$ N) in the direction of the 8 N force. Hence, the object cannot remain in equilibrium.

341. (a) $\Big[2^x + 3^x\Big]_0^1 = 5 - 2 = 3$
(b) $\Big[\log_2 x + \log_4 x\Big]_1^2 = \big(1 + \tfrac{1}{2}\big) - (0) = \tfrac{3}{2}$.

342. The equations are two concentric circles, whose radii are 1 and 2.



Since tangent and radius are perpendicular, we know by Pythagoras that half the chord has length

$$\tfrac{1}{2}l = \sqrt{2^2 - 1^2}.$$

Hence, the chord has length $l = 2\sqrt{3}$.

343. In terms of coins, we need to show that

$$\mathbb{P}(n+k \text{ heads}) = \mathbb{P}(n-k \text{ heads}).$$

If $n-k$ is the number of heads, then $n+k$ is the number of tails. So, the statement above is

$$\mathbb{P}(n+k \text{ heads}) = \mathbb{P}(n+k \text{ tails}).$$

Since heads and tails are equally likely, this is true by symmetry. □

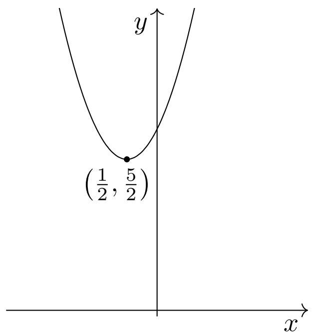
——— Alternative Method ———

Since this binomial distribution has probability $1/2$ and mean $2n \cdot 1/2 \equiv n$, it is symmetrical about $n$. The two probabilities in the required result are symmetrically $k$ away from $n$, so they must have equal probabilities. □

344. (a) Integrating f″, we get $f'(x) = 4x + c$. Then $f'(0)$ gives $c = 2$. So, $f'(x) = 4x + 2$.

(b) Integrating again, $f(x) = 2x^2 + 2x + d$. Then $f(0)$ gives $d = 3$. So, $f(x) = 2x^2 + 2x + 3$.

(c) The discriminant $\Delta = -20$ is negative, so the parabola doesn't cross the $x$ axis. Its vertex is $(1/2, 5/2)$ and its $y$ intercept is 3.




345. Since $x^2 - 1 \equiv (x - 1)(x + 1)$, we need to show that $(x - 1)$ and $(x + 1)$ are both factors of the cubic. So, we test $x = \pm 1$. Both of these values, when substituted into the cubic expression, give 0. Hence, $x^2 - 1$ is a factor, as required.

346. From $l = r\theta$, we rearrange to

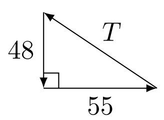
$$\frac{l}{2\pi r} = \frac{\theta}{2\pi}.$$

The LHS of the above is arc length $l$ as a fraction of circumference $2\pi r$. It is the scale factor from the circumference to the arc length. The same factor must also scale the area of the circle to give the area of the sector.

So, to find sector area, we multiply the area of the circle, which is $\pi r^2$, by the same fraction:

$$\begin{aligned} A &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \tfrac{1}{2} r^2 \theta, \text{ as required.} \end{aligned}$$

347. The object is in equilibrium, so the three forces, when placed tip-to-tail, form a closed triangle:




Pythagoras gives $T = \sqrt{48^2 + 55^2} = 73$.

348. Multiplying out and splitting up the fraction,

$$\begin{aligned} &\int \frac{(2x + 1)(2x - 1)}{x^2}\, dx \\ &= \int 4 - x^{-2}\, dx \\ &= 4x + \tfrac{1}{x} + c. \end{aligned}$$

349. We require each card to be one of the 40 non-face cards. Dealing one at a time without replacement,

$$\begin{aligned} p &= \tfrac{40}{52} \times \tfrac{39}{51} \times \tfrac{38}{50} \\ &= \tfrac{38}{85}. \end{aligned}$$

350. (a) Completing the square gives the result.

(b) The square is minimised at $x = -\frac{1}{2}p$, when it has value zero. At this point, the quadratic has value $q - \frac{1}{4}p^2$. So, $y = x^2 + px + q$ has a minimum at the given coordinates.

351. A function with a constant derivative is linear. Its output values at integer inputs therefore form an AP. Here, the common difference is 3. Counting explicitly, $h(2) = 7$ and $h(3) = 10$.

— Alternative Method —

From the two values given, the first derivative is $h'(x) = 3$. Integrating this, $h(x) = 3x + c$. Since $h(0) = 1$, $c = 1$. So, $h(x) = 3x + 1$, which gives $h(3) = 10$.

352. The factor $(x - a)$ changes sign at $x = a$. But an even power of it does not. The other factor is not zero at $x = a$, so it isn't relevant. Hence (a) and (c) change sign, but (b) doesn't.

— Nota Bene —

When considering a sign change, visualise moving (in $x$ terms) from one side of the relevant $x$ value to the other side. As this move happens, consider whether the output value $f(x)$ undergoes a sign change.

353. Multiplying out and simplifying,

$$\begin{aligned} & x - (\sqrt{x} - 2)^2 = 1 \\ \Longrightarrow\; & x - (x - 4\sqrt{x} + 4) = 1 \\ \Longrightarrow\; & 4\sqrt{x} - 4 = 1 \\ \Longrightarrow\; & x = \tfrac{25}{16}. \end{aligned}$$

354. (a) The cosine rule gives

$$\cos\theta = \frac{2^2 + 3^2 - 4^2}{2 \cdot 2 \cdot 3} = -\tfrac{1}{4}.$$

(b) The first Pythagorean trig identity is

$$\sin^2\theta + \cos^2\theta \equiv 1.$$

This gives $\sin^2\theta = 1 - \frac{1}{16} = \frac{15}{16}$. For non-reflex angles, $\sin\theta$ is positive, so $\sin\theta = \frac{1}{4}\sqrt{15}$.

(c) Using the value from part (b),

$$\begin{aligned} A &= \tfrac{1}{2} \cdot 2 \cdot 3 \cdot \tfrac{1}{4}\sqrt{15} \\ &= \tfrac{3}{4}\sqrt{15}. \end{aligned}$$

355. We solve as follows:

$$\begin{aligned} &|y^2 - 2| = 5 \\ \Longrightarrow\ & y^2 - 2 = \pm 5 \\ \Longrightarrow\ & y^2 = 7, -3. \end{aligned}$$

The latter produces no real roots as $y^2$ must be non-negative, so the solution is $y = \pm\sqrt{7}$.

356. (a) $4! = 24$.
(b) The number of orders of $\text{P}_1\text{P}_2\text{QR}$ is $4! = 24$ as in part (a). But $\text{P}_1$ and $\text{P}_2$ are in fact identical. So, we divide by an overcounting factor,which is the number of ways of ordering $\text{P}_1\text{P}_2$. This gives $4!/2! = 12$.

357. Integrating $f''(x)$ gives $f'(x) = 4x^{\frac{3}{2}} + c$. We know that $f'(0) = 5$, so $c = 5$. Hence $f'(x) = 4x^{\frac{3}{2}} + 5$.

358. Despite the unknown constants, this is a quadratic and can be factorised as usual:

$$\begin{aligned} &ax^2 + bx = 0 \\ \Longrightarrow\ & x(ax + b) = 0 \\ \Longrightarrow\ & x = 0, -\tfrac{b}{a}. \end{aligned}$$

359. (a) No, $x = -y$ is a possibility.
(b) Yes.
(c) Yes.
(d) Yes.

360. (a) $\frac{\pi}{6}$ radians,
(b) $9\pi$ radians.

361. (a) Yes: perpendicular to original line.
(b) No: parallel to original line.
(c) Yes, except if $a = b$: reciprocal gradient.

362. The eight small triangles around the outside are right-angled and isosceles. Their perimeters are 1, so their short sides have length $l$ satisfying

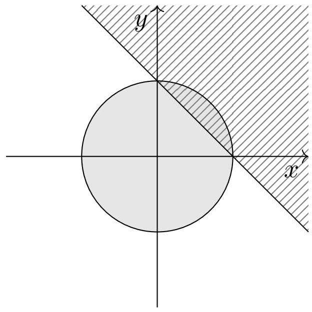
$$\begin{aligned} &2l + \sqrt{2}l = 1 \\ \Longrightarrow\ & l = \frac{1}{2 + \sqrt{2}}. \end{aligned}$$

Four of them together have area $2l^2$, which is $3 - 2\sqrt{2}$. The area of the octagon is then

$$\begin{aligned} A_{\text{octagon}} &= 1 - \left(3 - 2\sqrt{2}\right) \\ &= \sqrt{8} - 2. \end{aligned}$$

363. The bearings differ by 90°, so the paths the hikers take are at right-angles. Hence, the speed at which the hikers are separating is 5 mph, via a $(3, 4, 5)$ triple. So, it takes 1 hour.

364. Neither implication is true, as the line $x + y = 1$ and the circle $x^2 + y^2 = 1$ intersect twice.




The counterexamples are as follows:

(a) any $(x, y)$ point above and to the right of the line and inside the circle, i.e. doubly shaded,
(b) any $(x, y)$ point outside the circle and below and to the left of the line, i.e. not shaded.

365. The space diagonal is the long diagonal; it passes through the central space of the cuboid. Using 3D Pythagoras, its length is

$$\begin{aligned} d &= \sqrt{4^2 + 4^2 + 7^2} \\ &= 9. \end{aligned}$$

366. From the formula $s = ut + \frac{1}{2}at^2$, the magnitude of the acceleration must be $4\sqrt{2}$. Since this is in the direction of $\mathbf{i} + \mathbf{j}$, the acceleration is $4\mathbf{i} + 4\mathbf{j}$. So, the resultant force is $40\mathbf{i} + 40\mathbf{j}$. This means $\mathbf{F}_3 = 30\mathbf{i} + 20\mathbf{j}$.

367. The original curve is a positive parabola whose vertex is at $(a, b)$. The rotated curve is therefore a negative parabola whose vertex is also at $(a, b)$. Hence, its equation is $y = b - (x - a)^2$.

368. (a) Multiplying out,

$$\begin{aligned} &y = x^3 + x^2 - 12x \\ \Longrightarrow\ & \frac{dy}{dx} = 3x^2 + 2x - 12. \end{aligned}$$

(b) Rearranging to make $y$ the subject,

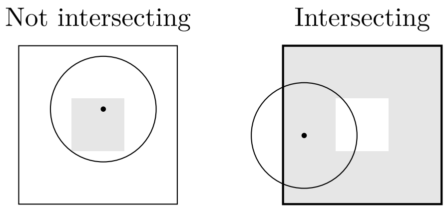
$$\begin{aligned} &y = x + 2x^{-1} \\ \Longrightarrow\ & \frac{dy}{dx} = 1 - 2x^{-2} \end{aligned}$$

369. To begin with, we solve for $x \in \mathbb{R}$. The boundary equation is $x^2 + 5x + 5 = 0$. The formula gives

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5}}{2} \\ &\approx -3.61, -1.38. \end{aligned}$$

The positive quadratic $x^2 + 5x + 5$ is less than zero between these roots. So, for $x \in \mathbb{Z}$, the solution is $x \in \{-3, -2\}$.

370. The interior of the square is the possibility space of locations for the centre of the circle. From the diagrams below, we can see that the total area is 9 and the area of the successful region is 8.




So the probability is $p = \frac{\text{successful}}{\text{total}} = \frac{8}{9}$.

— Nota Bene —

In this problem, the possibility space is continuous, as opposed to a discrete possibility space such as $\{1, 2, 3, 4, 5, 6\}$ which you can list. You can still use the formula $p = \frac{\text{successful}}{\text{total}}$; the concept carries through. Just consider "successful" and "total" as referring to continuous size of region (in this case area) rather than discrete number of outcomes.

371. Multiplying out and gathering like terms,

$$x^2 \equiv (A+B)x^2 + (A+B).$$

The coefficient of $x^2$ requires $A + B = 1$. But this forces the constant term to be 1, and there is no constant term on the LHS. So, this can never be an identity.

372. Substituting $x = 1$, the factor theorem requires

$$\begin{aligned} & k^2 - 12 + 2k^2 = 0 \\ \implies & k^2 = 4 \\ \implies & k = \pm 2. \end{aligned}$$

373. Integrating, $\text{f}(x) + c = \text{g}(x) + d$. The constants of integration can be combined: $\text{f}(x) = \text{g}(x) + k$. This tells us that the graphs $y = \text{f}(x)$ and $y = \text{g}(x)$ are a constant distance $k$ apart in the $y$ direction. □

374. The square has area 4, since each side is two radii. Subtracted from this are four quarter-circles, of total area $\pi$. So the area shaded is $4 - \pi$.

375. Call the odd numbers $2a+1$ and $2b+1$, for $a, b \in \mathbb{Z}$. Then their difference is $2a + 1 - (2b + 1)$, which is equal to $2(a - b)$. Since $a - b$ is an integer, this is an even number. QED.

— Nota Bene —

The technique of calling even numbers $2k$ and odd numbers $2k+1$, for some $k \in \mathbb{N}$, is a standard tool used in many proofs.

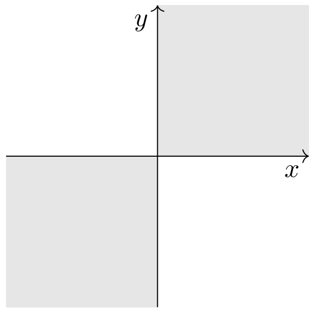
376. Differentiating to find the minimum, $14x - 9 = 0$. So, $x = \frac{9}{14}$. Substituting this gives

$$\text{f}(x) = \tfrac{255}{28} > 9.$$

Since the quadratic is positive, all outputs must therefore be greater than 9.

377. The implications in (a) and (c) are false for the same reason; a counterexample is $x = y = -1$, for which the RHS holds but the roots on the LHS are undefined. The implication in (b) is true, because negative numbers have real cube roots.

378. The boundary equation $xy = 0$ is solved by $x = 0$ or $y = 0$. These are the $x$ and $y$ axes. So the region in question is two quadrants: $x, y \geq 0$ and $x, y \leq 0$. The axes are included.




379. (a) The displacement is given by

$$\int_0^k 4t + 6 \, dt = \Big[2t^2 + 6t\Big]_0^k$$

Dividing this by the duration $k$ over which the displacement above takes place gives average velocity, which are told is 20 ms$^{-1}$.

(b) We get $2k + 6 = 20$, so $k = 7$.

380. Differentiating then integrating gives the original function, except with an unknown constant $c$ in place of the known constant. So,

$$\int \text{f}'(x) \, dx = 3x^2 - 2x + c.$$

381. Division by $\cos^2\theta$ gives

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 \equiv \frac{1}{\cos^2\theta}.$$

By definition, the first term on the LHS is $\tan^2\theta$ and the RHS is $\sec^2\theta$, giving the required result.

— Nota Bene —

This result is the second Pythagorean trig identity. It is used a little less often than the first. However, it still occurs regularly and should be thought of as a standard tool.

382. The minus sign shown below should be a plus, as the division line in a fraction acts like a bracket:

$$3x - 2 - x \underset{\uparrow}{-} 1 = 5x - 1.$$

383. To find the minimum, we set the first derivative to zero. This gives

$$\begin{aligned} -85 + 6n &= 0 \\ \implies n &= 14\tfrac{1}{6}. \end{aligned}$$

Since a quadratic is symmetrical around its vertex, $n = 14$ must give the minimum value, which is $-502$.

384. The object must be in equilibrium in both **i** and **j** directions, so $4a = 2b + 2$ and $b = a + 3$. Solving simultaneously gives $a = 4$, $b = 7$.

385. Since $a \propto b$ and $b \propto c$, we know that $a = k_1 b$ and $b = k_2 c$. Substituting gives $a = k_1 k_2 c$. This is $a \propto c$, with constant of proportionality $k_1 k_2$. QED.

386. The expression $\text{g}(2) - \text{g}(1)$ is the definite integral of $\text{g}'(x)$ between $x = 1$ and $x = 2$:

$$\begin{aligned} &\int_1^2 2x + 3\,dx \\ &= \Big[x^2 + 3x\Big]_1^2 \\ &= (4 + 6) - (1 + 3) \\ &= 6. \end{aligned}$$

387. The possibility space is the same in both cases. So, a king and a queen is likelier, as there are $4 \times 4 = 16$ successful outcomes, compared to ${}^4\text{C}_2 = 6$ for the two jacks.

388. A stretch factor 3 in the $y$ direction, followed by a translation by vector $2\mathbf{j}$.

389. The rectangular grid has area 8. The unshaded triangles have areas $2, 2, 1$. So, the shaded region has area 3.

390. We rearrange the first equation:

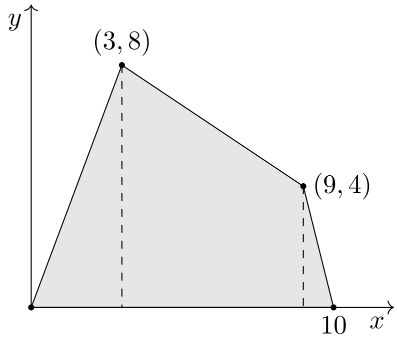
$$\begin{aligned} 0.1x + 0.3y &= 0.13 \\ \implies 30y &= 13 - 10x \\ \implies y &= \tfrac{13}{30} - \tfrac{1}{3}x. \end{aligned}$$

Substituting into the second equation gives

$$\begin{aligned} x^2 - 0.6\big(\tfrac{13}{30} - \tfrac{1}{3}x\big) &= 0.37 \\ \implies x^2 + \tfrac{1}{5}x - \tfrac{63}{100} &= 0 \\ \implies x &= -\tfrac{9}{10}, \tfrac{7}{10}. \end{aligned}$$

So, the solution points are $\big(-\frac{9}{10}, \frac{11}{15}\big)$, $\big(\frac{7}{10}, \frac{1}{5}\big)$.

391. Since two of the vertices of the quadrilateral are on the $x$ axis, it can be split up into three parts: two triangles and a trapezium.




The total area is

$$\begin{aligned} A &= A_{\text{triangle 1}} + A_{\text{trapezium}} + A_{\text{triangle 2}} \\ &= \tfrac{1}{2} \cdot 3 \cdot 8 + \tfrac{1}{2} \cdot 6 \cdot (8 + 4) + \tfrac{1}{2} \cdot 1 \cdot 4 \\ &= 50. \text{ as required.} \end{aligned}$$

392. Equating coefficients of **i** and **j**, we have a pair of linear simultaneous equations: $a + 2b = 11$ and $2a + b = 13$. The solution is $a = 5$, $b = 3$.

393. Since the $t$ values form an AP, we can treat the $x$ values as a sequence. For constant acceleration, we need the second differences (which correspond to the second derivative) to be constant. The first differences are $1, 3, 5$ and the second differences are $2, 2$. So, the data is consistent with the assumption of constant acceleration.

394. (a) Using the quadratic formula, if $\text{g}(x) = 0$, then

$$\begin{aligned} x &= \frac{649 \pm 649^2 - 4 \cdot 224 \cdot 255}{2 \cdot 224} \\ &= \frac{649 \pm 439}{448}. \end{aligned}$$

So, the roots are $x = \frac{17}{7}, \frac{15}{16}$.

(b) The factor theorem says that if $x = \frac{b}{a}$ is a root, then $(ax - b)$ is a factor. So, $(7x - 17)$ and $(16x - 15)$ are factors. Since $7 \times 16 = 112$, we also need a constant factor of two. Hence, $\text{g}(x)$ factorises as

$$224x^2 - 649x + 255 \equiv 2(7x - 17)(16x - 15).$$

395. Listing alphabetically:

| | |
|---|---|
| AAABB | ABBAA |
| AABAB | BAAAB |
| AABBA | BAABA |
| ABAAB | BABAA |
| ABABA | BBAAA |

396. The average value of the angles must be $\pi/3$. The smallest angle can be anything from zero to this average (since $\pi/3, \pi/3, \pi/3$ is trivially an AP). Hence, the set of possible values for the smallest angle is $(0, \pi/3]$.

397. (a) In 2D, any pair of non-parallel lines intersects, generating a simultaneous solution $(x, y)$.
   (b) For the lines to be parallel, $p = 1$.
   (c) For the lines to be distinct (otherwise they would have infinitely many solutions), $q \neq 10$.

398. The range is the set of output values attainable over the given domain. Every quadratic function produces a parabolic graph, which by definition has either a minimum or a maximum. Values one side of this are not attainable. Hence, there must be real numbers which are not in the range of any given quadratic function.

399. Factorising, $n^2 + n \equiv n(n+1)$. Since $n$ and $n+1$ are two consecutive integers, one of them must be even. Hence, their product must be even. $\square$

400. (a) This is false. An asteroid in deep space has mass but is (effectively) weightless.
   (b) This is true. In the Newtonian system, only massive objects experience gravitational force (weight).

——— END OF 4TH HUNDRED ———